JOURNAL OF COMBINATORIAL THEORY, Series A **75**, 141–144 (1996)
ARTICLE NO. 0067

Note

New Asymptotics for Bipartite Turán Numbers

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Communicated by the Managing Editors

Received March 3, 1994

An algebraic construction implies $\lim_{n\to\infty} \mathrm{ex}(n, K_{2,t+1})\, n^{-3/2} = \sqrt{t}/2$.

1. THE TURÁN PROBLEM

Given a graph F, what is $\mathrm{ex}(n, F)$, the maximum number of edges of a graph with n vertices not containing F as a subgraph? Until now, the only asymptotic for a bipartite graph which is not a forest, $\mathrm{ex}(n, C_4) = \frac{1}{2}(1+o(1))\, n^{3/2}$, is due to Erdős, Rényi and T. Sós [ERS] and (simultaneously and independently) to Brown [B].

THEOREM. *For any fixed* $t \geqslant 1$ $\mathrm{ex}(n, K_{2,t+1}) = \frac{1}{2}\sqrt{t}\, n^{3/2} + O(n^{4/3})$.

Let G be a graph on n vertices with e edges such that any two vertices have at most t common neighbors. The inequality $\sum_{x\in V}\binom{d(x)}{2} \leqslant t\binom{n}{2}$ (Kővári, T. Sós, Turán [KST]) implies the upper bound $e < \frac{1}{2}\sqrt{t}\, n^{3/2} + (n/4)$. To prove the Theorem we obtain a matching lower bound from a construction closely related to the examples from [ERS] and [B], and inspired by an example of Hyltén–Cavallius [H] and Mörs [M] given for Zarankiewicz's problem $z(n, n, 2, t+1)$ (see later in Section 3). The topic is so short of constructions that about 20 years ago, as a first step, Erdős [E67, E75] even proposed the problem whether $\lim_t(\liminf_n \mathrm{ex}(n, K_{2,t+1})\, n^{-3/2})$ goes to ∞ as $t \to \infty$. For a survey see Bollobás' book [Bo].

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0097-3165/96 $18.00

2. A Large Graph with no $K_{2,t+1}$

Let q be a prime power such that $(q-1)/t$ is an integer. We will construct a If $K_{2,t+1}$-free graph G on $n=(q^2-1)/t$ vertices such that every vertex has degree q or $q-1$. Then G has more than $(1/2)\sqrt{t}\,n^{3/2}-(n/2)$ edges. The lower bound for the Turán number for all n then follows from the fact for every sufficiently large n there exists a prime q satisfying $q\equiv 1 \pmod{t}$ and $\sqrt{nt}-n^{1/3}<q<\sqrt{nt}$ (see [HI]).

Let $\mathbf{F}$ be the q-element finite field, and let h be an element of order t, i.e., $h^t=1$ and the set $H=\{1, h, h^2, ..., h^{t-1}\}$ form a t-element subgroup of $\mathbf{F}\backslash\{0\}$. We say that $(a, b)\in\mathbf{F}\times\mathbf{F}$, $(a, b)\neq(0, 0)$ is equivalent to (a', b'), in notation $(a, b)\sim(a', b')$, if there exists some $h^\alpha\in H$ such that $a'=h^\alpha a$ and $b'=h^\alpha b$. The elements of the vertex set V of G are the t-element equivalence classes of $\mathbf{F}\times\mathbf{F}\backslash(0, 0)$. The class represented by (a, b) is denoted by $\langle a, b\rangle$. Two (distinct) classes $\langle a, b\rangle$ and $\langle x, y\rangle$ are joined by an edge in G if $ax+by\in H$. This relation is symmetric, and compatible to the equivalence classes, i.e., $ax+by\in H$, $(a, b)\sim(a', b')$, and $(x, y)\sim(x', y')$ imply $a'x'+b'y'\in H$.

For any given $(a, b)\in\mathbf{F}\times\mathbf{F}\backslash(0, 0)$ (say, $b\neq 0$) and for any given x and h^α, the equation $ax+by=h^\alpha$ has a unique solution for y. This implies that there are exactly tq solutions (x, y) with $ax+by\in H$. The solutions come in equivalence classes, one of these might coincide with $\langle a, b\rangle$ so the degree of the vertex $\langle a, b\rangle$ in G is either q or $q-1$.

We claim that G is $K_{2,t+1}$-free. First we show, that for $(a, b), (a', b')\in\mathbf{F}\times\mathbf{F}\backslash(0, 0)$, $(a, b)\nsim(a', b')$ the equation system

$$\begin{aligned} ax+by&=h^\alpha \\ a'x+b'y&=h^\beta \end{aligned} \tag{1}$$

has at most one solution $(x, y)\in\mathbf{F}\times\mathbf{F}\backslash(0, 0)$. Indeed, the solution is unique if the determinant $\det\begin{pmatrix} a & b \\ a' & b'\end{pmatrix}$ is not 0. Otherwise, there exists a c such that $a=a'c$ and $b=b'c$. If there exists a solution of (1) at all, then multiplying the second equation by c and subtracting it from the first one we get on the right hand side $h^\alpha-ch^\beta=0$. Thus $c\in H$, contradicting the fact that (a, b) and (a', b') are not equivalent.

Finally, there are t^2 possibilities for $0\leqslant\alpha, \beta<t$ in (1). The set of solutions again form t-element equivalence classes, so there are at most t classes $\langle x, y\rangle$ joint simultaneously to $\langle a, b\rangle$ and $\langle a', b'\rangle$.

The sets $N\langle a, b\rangle=\{\langle x, y\rangle: ax+by\in H\}$ almost form a $q+1$-uniform, symmetric t-design. This means that, they mutually intersect in exactly t elements except if $(a, b)=(ca', cb')$ holds for some c when they are disjoint. It seems to me that this structure, unfortunately, cannot be extended to a proper t-design.

3. Corollaries for Zarankiewicz's Problem

Given m, n, s and t, what is the maximum number, $z = z(m, n, s, t)$, such that there exists a $0-1$ matrix with m rows and n columns containing z 1's without a submatrix with s rows and t columns consisting of entirely of 1's. This question has become known as the problem of Zarankiewicz [Z]. For a bipartite graph F define the bipartite Turán number, $\mathrm{ex}(m, n, F)$, as the maximum number of edges in an F-free bipartite graph with m and n vertices in its color classes. Considering the adjacency matrix of a $K_{s,t}$-free graph on n vertices as a matrix of a bipartite graph on $n+n$ vertices (cf. [Bo] p. 310) we get

$$2\,\mathrm{ex}(n, K_{s,t}) \leqslant \mathrm{ex}(n, n, K_{s,t}) \leqslant z(n, n, s, t). \tag{2}$$

The determination of $z(m, n, s, t)$ is equivalent to a unidirectional Turán problem, when we label the two color classes of F and only those copies of F are forbidden in which the entire first color class is contained in the m-element set and the second color class lies in the n-element set.

It is easy to see that $z(n, n, 2, t+1) \leqslant n\sqrt{tn - t + 1/4} + (n/2)$, and it is known that this bound is asymptotically correct, i.e., $\lim_{n\to\infty} z(n, n, 2, t+1)\, n^{-3/2} = \sqrt{t}$ ([KST] for $t=1$, [H] for $t=2$ and [M] for all t). Our Theorem and the lower bound in (2) gives

Corollary. *For any fixed* $t \geqslant 1$ $\mathrm{ex}(n, n, K_{2,t+1}) = \sqrt{t}\, n^{3/2} + O(n^{4/3})$.

Thus we have a new near optimal construction for $z(n, n, 2, t+1)$. The gap between the lower and upper bounds in the case $n = (q^2 - 1)/t$ is only $O(\sqrt{n})$.

Acknowledgments

The author is indebted to M. Simonovits for fruitful conversations and Janice Malouf for helpful comments. This research was supported in part by the Hungarian NSF Grant OTKA 016389, and by NSA Grant MDA904-95-H-L045.

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